The Interaction Representation

we assume

$$H = H_0 + V$$

where V represents a small perturbation of the dynamics given by H_0;
a perturbation expansion with respect to V can be performed; the solution of the time–dependent Schrödinger equation reads

$$|\Psi(t)\rangle = U(t, t_0)|\Psi(t_0)\rangle$$

is conveniently written as

$$|\Psi(t)\rangle = U_0(t, t_0)|\Psi^{(\mathrm{I})}(t)\rangle$$

this representation makes use of the formal solution which is available for the unperturbed time–dependent Schrödinger equation for H_0, i.e.

$$U_0(t, t_0) = e^{-iH_0(t-t_0)/\hbar}$$

the new state vector $|\Psi^{(\mathrm{I})}(t)\rangle$ is called the state vector in the *interaction representation*;
since $U(t_0, t_0) = 1$ we have

$$|\Psi^{(\mathrm{I})}(t_0)\rangle = |\Psi(t_0)\rangle$$

the equation of motion for the state vector in the interaction representation follows directly from the original time–dependent Schrödinger equation,

$$i\hbar\frac{\partial}{\partial t}|\Psi(t)\rangle = U_0(t, t_0)\left(H_0|\Psi^{(\mathrm{I})}(t)\rangle + i\hbar\frac{\partial}{\partial t}|\Psi^{(\mathrm{I})}(t)\rangle\right) = H|\Psi(t)\rangle$$

after some rearrangement we get (note that $U^{-1} = U^{+}$)

$$i\hbar\frac{\partial}{\partial t}|\Psi^{(\mathrm{I})}(t)\rangle = U_0^{+}(t, t_0)VU_0(t, t_0)|\Psi^{(\mathrm{I})}(t)\rangle \equiv V^{(\mathrm{I})}(t)|\Psi^{(\mathrm{I})}(t)\rangle$$

the quantity $V^{(\mathrm{I})}(t)$ is the interaction representation of the perturbational part of the Hamiltonian;
the formal solution is obtained by introducing the so–called S–operator (the scattering matrix)

$$|\Psi^{(\mathrm{I})}(t)\rangle = S(t, t_0)|\Psi^{(\mathrm{I})}(t_0)\rangle \equiv S(t, t_0)|\Psi(t_0)\rangle$$

it follows

$$U(t, t_0) = U_0(t, t_0)S(t, t_0)$$

the S–operator can be determined by the iterative solution of the equation of motion; formal time–integration gives

$$|\Psi^{(\mathrm{I})}(t)\rangle = |\Psi^{(\mathrm{I})}(t_0)\rangle - \frac{i}{\hbar}\int\limits_{t_0}^{t} d\tau V^{(\mathrm{I})}(\tau)|\Psi^{(\mathrm{I})}(\tau)\rangle$$

this equation is suited to develop a perturbation expansion with respect to $V^{(\mathrm{I})}$. If there is no interaction one gets

$$|\Psi^{(\mathrm{I},0)}(t)\rangle = |\Psi^{(\mathrm{I})}(t_0)\rangle$$

next we get the state vector in the interaction representation, which is the first–order correction to $|\Psi^{(\mathrm{I},0)}(t)\rangle$ in the presence of a perturbation,

$$|\Psi^{(\mathrm{I},1)}(t)\rangle = -\frac{i}{\hbar}\int\limits_{t_0}^{t} d\tau_1 V^{(\mathrm{I})}(\tau_1)|\Psi^{(\mathrm{I},0)}(\tau_1)\rangle$$

upon further iteration of this procedure one obtains the nth–order correction as

$$|\Psi^{(\mathrm{I},n)}(t)\rangle = -\frac{i}{\hbar}\int\limits_{t_0}^{t} d\tau_n V^{(\mathrm{I})}(\tau_n)|\Psi^{(\mathrm{I},n-1)}(\tau_n)\rangle$$

the total formally exact state vector in the interaction representation is

$$|\Psi^{(\mathrm{I})}(t)\rangle = \sum_{n=0}^{\infty}|\Psi^{(\mathrm{I},n)}(t)\rangle$$

let us consider the total wave function containing the effect of the interaction up to the order n

$$|\Psi^{(\mathrm{I},n)}(t)\rangle = \left(-\frac{i}{\hbar}\right)^n \int\limits_{t_0}^{t} d\tau_n V^{(\mathrm{I})}(\tau_n) \int\limits_{t_0}^{\tau_n} d\tau_{n-1} V^{(\mathrm{I})}(\tau_{n-1}) \times \ldots$$

$$\ldots \times \int\limits_{t_0}^{\tau_2} d\tau_1 V^{(\mathrm{I})}(\tau_1)|\Psi^{(\mathrm{I})}(t_0)\rangle$$

$$= \left(-\frac{i}{\hbar}\right)^n \frac{1}{n!}\,\hat{T}\int\limits_{t_0}^{t} d\tau_n \ldots d\tau_1 V^{(\mathrm{I})}(\tau_n)\ldots V^{(\mathrm{I})}(\tau_1)|\Psi^{(\mathrm{I})}(t_0)\rangle$$

in the last part of this expression all integrals are carried out to the upper limit t; double counting is compensated for by the factor $1/n!$; in order to account for the fact that the time–dependent operators $V^{(\mathrm{I})}$ do not commute for different time arguments the *time ordering operator* $\hat{T}$ has been introduced; it orders time–dependent operators from the right to the left with increasing time arguments, i.e., if $t_1 > t_2$, $\hat{T}[V^{(\mathrm{I})}(t_2)V^{(\mathrm{I})}(t_1)] =$

$V^{(\mathrm{I})}(t_1)V^{(\mathrm{I})}(t_2)$;

this formal rearrangement enables us to write for the exact state vector in the interaction representation

$$|\Psi^{(\mathrm{I})}(t)\rangle = \hat{T}\sum_{n=0}^{\infty}\frac{1}{n!}\prod_{k=1}^{n}\left(-\frac{i}{\hbar}\int\limits_{t_0}^{t} d\tau_k V^{(\mathrm{I})}(\tau_k)\right)|\Psi^{(\mathrm{I})}(t_0)\rangle$$

the summation on the right–hand side is formally identical to the expansion of the exponential function

$$S(t,t_0) = \hat{T}\ \exp\left\{-\frac{i}{\hbar}\int\limits_{t_0}^{t} d\tau V^{(\mathrm{I})}(\tau)\right\}$$